Quantum mechanical Renormalization Group and Multigrid

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based on joint work with B. Feddersen

References

[1] Cohen Tannoudji et al, *Electron Photon Interactions*, ch. III.2

[2] V. Bach, J. Fröhlich, I.M. Sigal, *Renormalization Group Analysis of Spectral Problems in Quantum Field Theory*, preprint SFB 288 TU Berlin

[3] W. Hackbusch, *Iterative Lösung großer schwachbesetzter Gleichungssysteme*, Teubner Stuttgart 1993

Effective Hamiltonians in QM

Given a Hamiltonian $H : \mathcal{H} \mapsto \mathcal{H}$ in a Hilbert space $\mathcal{H}$, suppose one is interested in states with eigenvalues near E. Consider

$$H[z] = H - z, \quad z \in D$$

where D is a complex neighbourhood of E. Given projectors P ($P^2 = P$, not necessarily selfadjoint), $Q = 1 - P$, such that QHQ has no spectral values in D, so that the "high frequency propagator" on $Q\mathcal{H}$,

$$\Gamma[z] = Q\,(QHQ - z)^{-1}\,Q$$

exists, one defines an effective (z-dependent) Hamiltonian

$$\begin{aligned} H[z]_{eff} &: \quad P\mathcal{H} \mapsto P\mathcal{H} && (1)\\ H[z]_{eff} &= \quad P\,(H - z)\,P - PH\Gamma[z]HP. && (2) \end{aligned}$$

Feshbach-projection [2], *Schur complement* [3].

Properties

Eigenvalues of H inside D are exactly those $z \in D$ for which $H[z]_{eff}$ has no inverse.

Inverse : If z is not an eigenvalue,

$$(H-z)^{-1} = (1-\Gamma[z]HP)\,(H[z]_{eff})^{-1}\,(1-PH\Gamma[z])$$

Eigenfunctions If $H[E]_{eff}\varphi = 0$ for some E and some $\varphi \in P\mathcal{H}$, then

$$\Psi = (1-\Gamma[E]QH)\,\varphi$$

obeys $(H-E)\Psi = 0$.

The problem is reduced to the study of $H[z]_{eff}$, modulo mastery of the "high frequency problem" (determination of Γ.)

Note: $H[z]_{eff}$ is a nonlinear function of z, but holomorphic in D. It is also a *nonlinear function of H* , in this sense there is *nontrivial renormalization.*

Typical applications

$$H = H_0 + V, \quad [P, H_0] = 0,$$

i.e.y project on eigenvalues of H_0 near E. In favorable cases [2], Γ can be computed by *convergent perturbation theory.*

Demonstration à la Wigner

(i.e. with 2×2 matrices.)

$$H - z = \begin{pmatrix} -z & v \\ v & 1-z \end{pmatrix} = H_0 + V - z \;,\; P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3)$$

Eigenvalues: solutions of $-z(1-z) - v^2 = 0$,

$$z_{\pm} = \frac{1}{2}\left(1 + \sqrt{1+4v^2}\right) \approx -v^2,\ 1+v^2. \quad (4)$$

Compute $\Gamma[z] = Q[1-z]^{-1}Q$, $P(H_0 - z)P = -zP$

$$\begin{aligned} H[z]_{eff} &= P\left[-z - v[1-z]^{-1}v\right]P \\ &= [1-z]^{-1}P\left(-z(1-z) - v^2\right)P \end{aligned}$$

Inverse nonexistent when $(\ldots) = 0$.

Renormalization group usage

Suppose there is a bijective *rescaling map*

$$\sigma : P\mathcal{H} \mapsto \mathcal{H}$$

Then

$$\sigma H[z]_{eff}\sigma^{-1} : \mathcal{H} \mapsto \mathcal{H}$$

and one has a comparable problem to that defined by $H[z]$.

Define *fixed points*, *relevant*, *marginal*, and *irrelevant perturbations*, etc. And so it goes ... [2].

Multigrid

Seek solution $u = L^{-1}f$ of

$$Lu = f$$

L = Dirac operator (for instance), $f, u \in \mathcal{H}$ = space of (vector valued) functions on a grid Λ.

Multigrid method:

$\mathcal{H}'$ = space of functions on a coarse grid Λ'.

averaging (restriction) operator : $\mathcal{C} : \mathcal{H} \mapsto \mathcal{H}'$

interpolation (prolongation) operator : $\mathcal{A} : \mathcal{H}' \mapsto \mathcal{H}$

Assume $\mathcal{C}\mathcal{A} = 1$. Then $P = \mathcal{A}\mathcal{C}$ obeys $P^2 = P$. Set $H = L$.

$$\begin{aligned} H[0]_{eff} &= \mathcal{A} L_{eff} \mathcal{C} \\ L_{eff} &= L^0_{eff} - \mathcal{C} L \Gamma L \mathcal{A} \qquad (5) \\ L^0_{eff} &= \mathcal{C} L \mathcal{A} \,, \quad \Gamma = \mathcal{A} (PLP)^{-1} \mathcal{C} \qquad (6) \end{aligned}$$

(inverse on $P\mathcal{H}$). Define *ideal interpolation and averaging operators* (for given $\mathcal{C}$)

$$\begin{aligned} \tilde{\mathcal{A}} &= (1 - \Gamma L) \mathcal{A} \qquad (7) \\ \tilde{\mathcal{C}} &= \mathcal{C} (1 - L\Gamma) \qquad (8) \end{aligned}$$

Then

$$L_{eff} = \mathcal{C}L\tilde{\mathcal{A}} = \tilde{\mathcal{C}}L\mathcal{A} \tag{9}$$

$$L^{-1} = \tilde{\mathcal{A}}L_{eff}^{-1}\tilde{\mathcal{C}} + \Gamma \tag{10}$$

Therefore $Lu = f$ is solved by

$$u = \Gamma f + \tilde{\mathcal{A}}U \tag{11}$$

$$L_{eff}U = F \quad \text{where } F = \tilde{\mathcal{C}}f. \tag{12}$$

Under the assumptions that PLP has no spectral value near 0, Γf can be computed by fast converging iteration, and so can L_{eff} and $\tilde{\mathcal{A}}U$ and $\tilde{\mathcal{C}}f$.

Problem: *The exact L_{eff} has exponential tails and is therefore unsuitable for practical calculation.* $L_{eff}U$ is practically computable, though.

Solution in principle: Approximate $L_{eff} \approx L'$ with manageable L'. Then iterate

$$L'U_{new} + (L_{eff} - L')\,U_{old} = F. \tag{13}$$

Convergence will be fast if $||L'^{-1}(L_{eff} - L')||$ is small. One needs to approximate the low lying spectrum well!

We are back to the problem of approximating effective Hamiltonians...

Bäkers approach amounts to approximating $\tilde{\mathcal{A}} \approx \mathcal{A}'$ and computing $L' = \mathcal{C}L\mathcal{A}'$.

Alternative: The results on the low lying spectrum of the Dirac operator (see S. Meyers talk) suggest a scaling

$$\not{D}_{eff} = Z \not{D}\,' + \text{irrelevant} \tag{14}$$

For staggered fermions, the irrelevant terms ought to be small. For Wilson fermions a Sheikholeslamy Wohlert correction term may be needed,

$$\text{irrelevant} = \gamma \not{D}\,'^2 + \gamma' \Delta + \kappa \mathbf{1} + \ldots$$

$\not{D}\,'$ involves a (unitary) blocked gauge field, possibly with a nontrivial β-dependent scaling Z. The iterative improvement (13) makes the method exact.