Progress
with the Real Space Renormalization Group

Gerhard Mack

DFG Kolloquium Wuppertal, 21 Feb. 1997

based on joint work with

M. Griessl, G. Palma, Y. Xylander, NPB **B 477** 878 (1996)

and in preparation (with M. Bartels)

Practical Relevance of Real Space RG-calculations

Exact effective lattice action and observables $\Rightarrow$ exact continuum results from a lattice action.

perfect or improved actions are approximations to exact effective lattice actions.

It is now universally agreed that the use of improved actions is crucial to get accurate quantitative results from lattice simulations especially for theories with fermions.

Wilsons real space RG: Compute effective actions by integrating out high frequency modes of the fields

Problems:

1. Parametrization of the effective action to make it manageable
2. Doing the integration

By taking small step size in length scale, perturbation theory can be made accurate. Therefore
the crucial problem is the parametrization.

Expansion around classical perfect actions

original lattice Λ: points z, field $\phi(z)$, action $S(\phi)$

is mapped to

block lattice Λ': points x, field $\Phi(x)$, action $S_{eff}(\Phi)$

by a block spin transformation

$$\Phi(x) = C\phi(x)$$

C = some average over blocks, e.g. (linear)

$$C\phi(x) = \underset{z\in x}{av}\ \phi(z) \equiv \mathcal{C}\phi(x).$$

effective action

$$\exp\left(-S_{eff}(\Phi)\right) = \int D\phi \prod_x \delta(\Phi(x) - C\phi(x)) \exp\left(-S(\phi)\right)$$

Proceed in two steps (Balaban)

- Compute the classical perfect action

$$S_{eff,cl}(\Phi) = S(\Psi[\Phi]) = max\ S(\phi)$$

subject to constraint $C\phi = \Phi$.

background field = maximizing field $\Psi = \Psi[\Phi]$

- Compute loop corrrections by expanding around background field, e.g. (linear)

$$\phi(z) = \Psi(z) + \zeta(z), \quad \mathcal{C}\zeta = 0$$

ζ = fluctuation field (high frequency field)

$$S(\phi) = S(\Psi) + \frac{1}{2}\zeta S''(\Psi)\zeta + \ldots$$

1-loop calculation yields

$$\begin{aligned} S_{eff}(\Phi) &= \tilde{S}_{eff}(\Psi) = S(\Psi) - \frac{1}{2} tr \ln \Gamma[\Psi] \qquad (1) \\ \Gamma[\Psi] &= \lim_{\kappa \mapsto \infty} \left(S'' + \kappa \mathcal{C}^* \mathcal{C}\right)^{-1} \end{aligned}$$

The effective action is especially simple when considered as a function of the background field:

equal to original action in class approx.

For nonlinear $\sigma - model$ the 1-loop correction merely changes overall factor (coupling constant) β, cp. below.

Parametrization of the action

in a way which is preserved by the RG-flow. One only needs $S(\phi)$ for smooth fields.

Observations:

- locality: $S_{,z}(\phi) \equiv \frac{\delta}{\delta\phi(z)} S(\phi)$ depends on $\phi(w)$ only in a neighbourhood of z of ≈ 1 fine lattice spacing.
- $\Psi[\phi](z)$ is a smooth interpolation of Φ, therefore nearly constant over such distances
(for large enough scale factor)
- $\Gamma[\Psi]$ can be computed from knowledge of $S''(\Psi)$.

Conclusion from (1): To iterate RG-step it suffices to know $S''(\bar{\phi})$ for constant field $\bar{\phi}$. If $\bar{\phi}$ has N components,

the action gets parametrized by two (vector resp. matrix) functions $S_{,z}(\bar{\phi}), S_{,zw}(\bar{\phi})$ of N real variables.

For some models, symmetry restricts them to $\bar{\phi}$-independent constants, e.g. S_{zw}= "perfect Laplacian".

no polynomial approximation in the field is made. In $\tilde{S}(\Psi)$, higher orders of $(\nabla\Psi)$ than 2nd are neglected, but **not** higher derivatives $\nabla^n\Psi$

Ingredients of the perturbative computation

Need high frequency propagator $\Gamma[\Psi]$
and interpolation operator $\mathcal{A}(z,x|\Psi) = \frac{\delta}{\delta\Phi(x)}\Psi[\Phi](z)$

For free fields determined by Kupiainen + Gawedzki.

It suffices to know their derivatives for constant fields.
There exist formulas for derivatives.

Computer assisted calculation from Fourier transforms

Results for ϕ^4-theory in 2 dimensions

Results for $S'(\bar{\phi})$ are compared with computer simulations. They are accurate also for large field $\bar{\phi}$ where perturbation theory with polynomial approximation fails.

$O(N+1)$-symmetric nonlinear σ-model in 2 dimensions

original action on Λ^0:$\phi(z) \in S^N$

$$S^0 = \beta \int_z \nabla_\mu \phi(z) \nabla_\mu \phi(z) \ ,$$

Block Spin: nonlinear

$$\Phi(x) = \mathcal{C}\phi(x)/|\mathcal{C}\phi(x)|, \quad (2)$$

$$\mathcal{C}\phi(x) = \underset{z \in x}{av} \ \phi(z) \quad (3)$$

equivalent definition: $\Phi(x)$ is the unit vector such that the block average of the components of $\phi(z)$ perpendicular to $\Phi(x)$ vanishes.

background field $\Psi[\Phi](z)$

determined by the block spin $\Phi(x)$:

$$S^i(\Psi) = max$$

subject to the condition

$$C\Psi(x) = \Phi(x)$$

Decomposition of the field $\phi(z)$ into background field $\Psi(z)$ and fluctuation field $\zeta(z)$:

Trick: parametrize the field so that the block spin constraint $C\phi = \Phi$ becomes a linear constraint on ζ,

$$\mathcal{C}\zeta(x) \equiv \underset{z\in x}{av}\ \zeta(z) = 0$$

This is achieved by choosing

$$\zeta(z) \in T_{\Phi(x)}S^N, \text{ i.e. } \zeta(z) \perp \Phi(x)$$

for $z \in x$. Let $\Psi^\perp$= component $\perp \Phi(x)$.

$$\begin{aligned} \phi^\perp(z) &= \Psi^\perp(z) + \zeta(z) && (4)\\ \phi^{||}(z) &= \left[1 - |\phi^\perp(z)|^2\right]^{1/2} . && (5) \end{aligned}$$

The effective action becomes (with a Jacobian J)

$$\exp\left(-S_{eff}(\Phi)\right) = J(\Psi)\int \prod_z d^N\zeta(z)\prod_x \delta^{(N)}(\mathcal{C}\zeta(x))$$
$$\exp\left(S(\Psi)+\frac{1}{2}\zeta S''(\Psi)\zeta + ...\right) \quad (6)$$
$$= J(\Psi)\exp\left(-\tilde{S}_{eff}(\Psi)\right) \;, \quad (7)$$
$$\delta(\mathcal{C}\zeta(x)) = \lim_{\kappa\mapsto\infty} N_\kappa \exp\left(-\zeta\mathcal{C}^*\mathcal{C}\zeta\right). \quad (8)$$

This is a Gaussian integral!

Transformation to Polyakov basis

Next transform to a z-dependent new orthonormal basis

$$\mathbf{e}^0(z) = \Psi(z)$$
$$\mathbf{e}^1(z) = [sin\theta]^{-2}\left[\Phi(x) - \Psi(z)\Psi(z)\cdot\Phi(x)\right] \quad (9)$$
$$\mathbf{e}^a(z) \qquad (a = 2...N) \text{ arbitrary}$$
$$cos\theta \equiv \Psi(z)\cdot\Phi(x),$$

and to variables $\xi \in T_{\Psi(z)}S^N$, i.e. $\xi \perp \Psi(z)$,

$$\xi^1 = \zeta^1/cos\theta, \quad \xi^i = \zeta^i \; (i = 2...N)$$

The exponent becomes

$$\zeta S''(\Psi)\zeta = \beta \int_z (\nabla_\mu \xi)^2 + \xi^i A^0{}_{i\mu} A_{0j\mu} \xi^j - \xi^2 A^i{}_{0\mu} A_{i0\mu}$$
$$+\frac{1}{2} tan\theta \xi^2 \nabla_\mu A_{10\mu} + ...$$
$$A^i_{0\mu} = \mathbf{e}^i \nabla_\mu \Psi.$$

Result after performing the Gaussian integration

$$\tilde{S}_{eff}(\Psi) = \int_z \beta_{eff} (\nabla \Psi(z))^2 + \frac{1}{2} \int_z \Gamma(zz) \tan\theta(z) \mathbf{e}^1 \Delta \Psi(z)$$
$$\beta_{eff} = \beta - (N-1)\Gamma(zz), \quad (10)$$

Correction term (2nd term) is small for smooth Φ,

the 1-loop perfect action equals the classical perfect action, except for the running of the coupling.

The same kind of computation is feasible for gauge theories

cp. U.Kerres, G. Mack, G. Palma, NPB (1996),
Diss U. Kerres Hamburg 1996

Hasenfratz and Niedermayer conjectured that there the classical perfect action is also 1-loop perfect. Correction term ...

Effective action $S_{eff}(\Phi)$

For smooth fields Φ the extremizing background field $\Psi[\Phi]$ can be determined by iteration to the accuracy needed for

iterating the RG-transformation. Result:
The effective action is a polynomial of 4-th order in the spins, with analytically determined coefficients.

It involves "energy correlations"

$$\beta_{eff}^{-1} S_{eff}(\Phi) = \int \Phi \underline{\mathbf{\Delta}} \Phi - \Phi(\mathcal{C} - \mathcal{A}^*)\mathcal{L}(\mathcal{C}^* - \mathcal{A})\Phi - \mathcal{L}\Gamma\mathcal{L}$$

$$\mathcal{L}(z) = \Phi(z)\Delta\Phi(z), \quad (11)$$

$$\Phi(z) = \Phi(x) \text{ for } z \in x. \quad (12)$$

Γ = Kupiainen Gawedzki high frequency propagator
$\mathcal{A}$ = KG interpolation operator
$\underline{\mathbf{\Delta}}$ = perfect Laplacian (of KG).

Towards computer assisted analytical solution of QCD:

What could be the

effective low energy action for QCD?

A true low energy effective action ought to give accurate results in tree approximation

Expressed in terms of background fields the action could be very simple - but its tree approximation is not trivial!

Conjecture (which we would like to derive by RG):

Effective dielectric theory

Variables: Besides gauge fields and Dirac matter fields, a tensorial dielectric field

$$\epsilon_{\mu\nu\rho\sigma}(z) \approx \sigma_\mu(z)\sigma_\nu(z)\delta_{\mu\rho}\delta_{\nu\sigma}$$

Action contains a potential $V(\epsilon)$ whose minimum is at 0.

It was proven for a model that this leads to flux tubes and to confinement, because

$$\nabla D = 4\pi\rho, \tag{13}$$

$$D = \epsilon E. \tag{14}$$

can have a solution for $\rho \neq 0$ only for $\epsilon \neq 0$.

Analogy: Nonlinear sigma-model: After some flow to larger coupling, the action as a function of unit-spins becomes nonlocal, and one needs block spins of fluctuating lengths $\phi = \epsilon \cdot$unit-vector.
The action acquires a potential term $V(\epsilon^2)$ which becomes convex eventually, leading to mass generation.